Sept 21, 2022

Week 3

LL

We knew how to evaluate $\iint_R f$ where R is a rectangle. We'd like to define

$$\iint_D f$$

where D is a region.

A <u>region</u> | <u>domain</u> is the set bounded by one or several curves. It consists of interior points and boundary points.

examples of regions.

Given a function on a set $E \subset \mathbb{R}^2$, we extend it to be a function in $\mathbb{R}^2$ by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & , (x,y) \in E \\ 0 & , (x,y) \notin E. \end{cases}$$

Given a function f defined on a region D. We pick a rectangle R containing D and define

$$\iint_D f = \iint_R \tilde{f}.$$

We point out

- If f is piecewise continuous on D, $\tilde{f}$ is piecewise continuous in $\mathbb{R}^2$, so $\tilde{f}$ is integrable in any R.

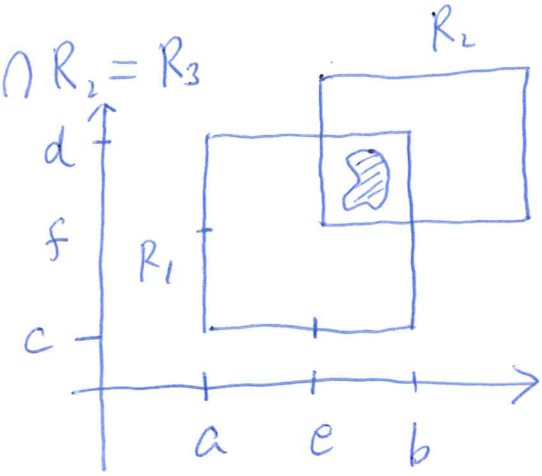
- If $D \subset R_1$, $D \subset R_2$,

$$\iint_{R_1} \tilde{f} = \iint_{R_2} \tilde{f},$$

hence the above definition is indept of the choice of R containing D.

▨ Pf: Let $R_1 = [a,b] \times [c,d]$, $R_1 \cap R_2 = R_3$

$R_3 = [e,b] \times [f,d]$.



Consider a partition P on R_1 taking e and f as endpoints. Then all subrectgles of P inside R_3 form a partition of R_3. Denote it by P'.

Consider the Riemann sum

$$S(\tilde{f},P) = \sum_{P} \tilde{f}(P_{ij}) \Delta x_i \Delta y_j$$

$$= \sum_{P'} \tilde{f}(P_{ij}) \Delta x_i \Delta y_j + \sum_{\substack{\text{outside} \\ P'}} \tilde{f}(P_{ij}) \Delta x_i \Delta y_j .$$

Pick P_{ij} to be the center of R_{ij},
then $\tilde{f}(P_{ij}) = 0$ for R_{ij} not inside P'. therefore,

$$S(\tilde{f},P) = \sum_{P'} \tilde{f}(P_{ij}) \Delta x_i \Delta y_j$$

$$= S(\tilde{f},P')$$

As $\|P\| \to 0$, LHS $\to \iint_{R_1} f$, RHS $\to \iint_{R_3} f$

We conclude $\iint_{R_1} \tilde{f} = \iint_{R_3} \tilde{f}$.

Similarly, can show

$$\iint_{R_2} \tilde{f} = \iint_{R_3} \tilde{f}$$

$\therefore \iint_{R_1} \tilde{f} = \iint_{R_2} \tilde{f}$. ▨

When D is described as

$$\{(x,y) : g_1(x) \le y \le g_2(x),\ a \le x \le b\},$$

then

$$\iint_D f = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

▨ Pf:

$$\iint_D f \overset{\text{def}}{=} \iint_R \tilde{f}, \qquad D \subset R = [a,b]\times[c,d].$$

$$= \int_a^b \left(\int_c^d \tilde{f}(x,y)\,dy \right) dx \qquad \text{(Fubini's thm)}$$

$$= \int_a^b \left(\int_c^{g_1(x)} \tilde{f}(x,y)\,dy + \int_{g_1(x)}^{g_2(x)} \tilde{f}(x,y)\,dy + \int_{g_2(x)}^{d} \tilde{f}(x,y)\,dy \right) dx$$

$$= 0 + \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx + 0,$$

since $\tilde{f}(x,y) = 0$ when $c \le y \le g_1(x)$, $\tilde{f}(x,y) = 0$ when $g_2(x) \le y \le d$, and $\tilde{f}(x,y) = f(x,y)$ when $g_1(x) \le y \le g_2(x)$, $\forall x \in [a,b]$.

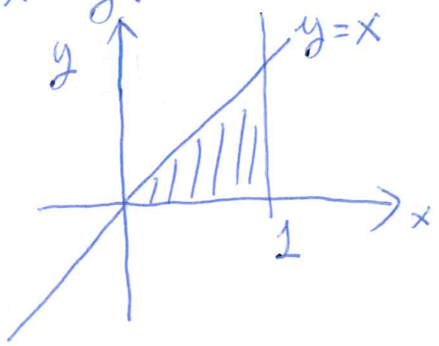
▨

Switching x and y, where

$$D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\},$$

$$\iint_D f = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy .$$

e.g. Find the volume of the prism whose base is the triangle in xy-plane bdd by $y=x$, $x=1$, and the x-axis, and whose top is $z = 3-x-y$.



the triangle can be described as

$$T = \{(x,y) : 0 \le y \le x,\ 0 \le x \le 1\}$$

$$\therefore \text{volume} = \iint_T (3-x-y)\,dA(x,y)$$

$$= \int_0^1 \int_0^x (3-x-y)\,dy\,dx = \int_0^1 \left(3y - xy - \frac{y^2}{2}\right)\Big|_0^x dx$$

$$= \int_0^1 \left(3x - \frac{3}{2}x^2\right) dx = \left(3\frac{x^2}{2} - \frac{3}{2}\frac{x^3}{3}\right)\Big|_0^1 = 1 .$$

Alternatives,

$$T = \{(x,y) : y \le x \le 1,\ 0 \le y \le 1\}$$

$$\therefore \text{volume} = \iint_T (3-x-y)\,dA(x,y)$$

$$= \int_0^1 \int_y^1 (3-x-y)\,dx\,dy = \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy$$

$$= \left(\frac{5}{2}y - 2y^2 + \frac{3}{2}\frac{y^3}{3}\right)\Big|_0^1 = 1.$$

e.g. Evaluate

$$\iint_T \frac{\sin x}{x}\, dA$$

where T is the triangle in the previous example.

$$\iint_T \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \frac{\sin x}{x}\, x\, dx = -\cos x\Big|_0^1$$

$$= -\cos 1 + 1 \approx 0.46.$$

On the other hand,

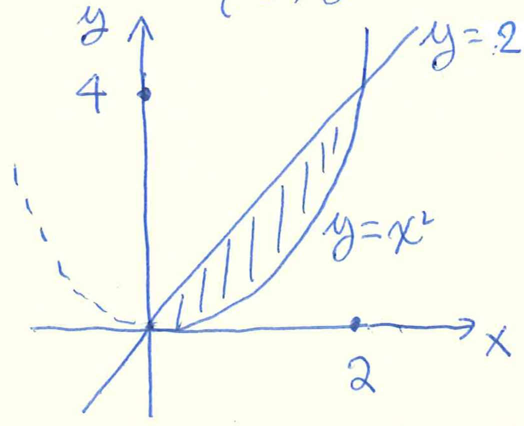
$$\iint_T \frac{\sin x}{x}\, dA = \int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$$

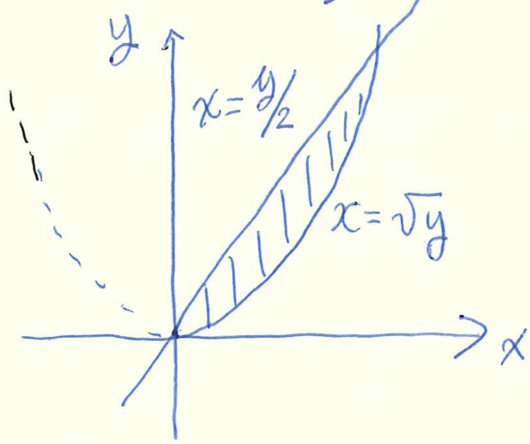
can't be done. So the order of integration matters!

e.g. $\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx$. Change the order of integration.

The region is $\{(x,y):\ x^2 \le y \le 2x,\ 0 \le x \le 2\}$

ie.





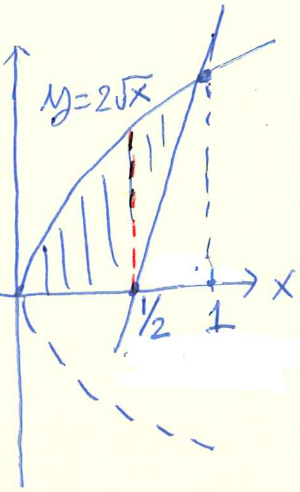
The region can be described as

$$\{(x,y) : \sqrt{y} \le x \le y/2 \ , \ 0 \le y \le 4\}.$$

Hence

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy .$$

e.g. Find the volume of the solid lying beneath $z = 16 - x^2 - y^2$ and above the region D bounded by $y = 2\sqrt{x}$, $y = 4x-2$, and the x-axis.



D can be decomposed into

$$D_1 = \{(x,y) : 0 \le y \le 2\sqrt{x} \ , \ 0 \le x \le \tfrac{1}{2}\}$$

and

$$D_2 = \{(x,y) : 4x-2 \le y \le 2\sqrt{x} \ , \ \tfrac{1}{2} \le x \le 1\}$$

Hence

$$\iint_D (16-x^2-y^2)\,dA = \iint_{D_1} (16-x^2-y^2)\,dA + \iint_{D_2} (16-x^2-y^2)\,dA$$

$$= \int_0^{\frac{1}{2}} \int_0^{2\sqrt{x}} (16-x^2-y^2)\,dy\,dx + \int_{\frac{1}{2}}^{1} \int_{4x-2}^{2\sqrt{x}} (16-x^2-y^2)\,dy\,dx$$

$$\vdots$$

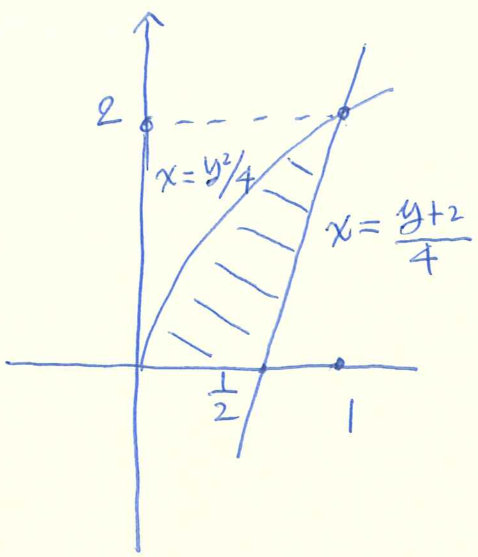
$$= \frac{20803}{1680} \approx 12.4 .$$

On the other hand, D can be described as

$$D = \{(x,y) : y^2/4 \le x \le \frac{y+2}{4} \ , \ 0 \le y \le 2\}$$

$$\therefore \iint_D (16-x^2-y^2)\, dA = \int_0^2 \int_{y^2/4}^{(y+2)/4} (16-x^2-y^2)\, dx\, dy$$

$$= \frac{20803}{1680}.$$



Three basic properties of integrals.

(linearity) Ⓐ $\iint_D (\alpha f + \beta g)\, dA = \alpha \iint_D f + \beta \iint_D g\, dA, \ \forall \alpha, \beta \in \mathbb{R}.$

(positivity) Ⓑ $f \geq 0$ on $D \Rightarrow \iint_D f\, dA \geq 0$

(decomposition) Ⓒ Let a curve C divide D into 2 regions D_1 and D_2. Then for continuous f,

$$\iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$$

(A) and (B) can be proved easily by looking at Riemann sums.

(C) can be deduced from (A). Need a basic concept.

For any non-empty set E in $\mathbb{R}^2$, its <u>characteristic function</u> χ_E is a function in $\mathbb{R}^2$ given by

$$\chi_E(x,y) = \begin{cases} 1, & (x,y) \in E, \\ 0, & (x,y) \notin E. \end{cases}$$

We've the formula

$$\chi_{E \cup F} = \chi_E + \chi_F - \chi_{E \cap F}$$

(see Ex 2).

▨ Pf of (C). Use $D_1 \cup D_2 = D$ and $C = D_1 \cap D_2$.

$$\chi_D = \chi_{D_1} + \chi_{D_2} - \chi_C$$

from the above formula. Let $D \subset R$.

$$\iint_D f = \iint_R \tilde{f}\chi_D = \iint_R (\tilde{f}\chi_{D_1} + \tilde{f}\chi_{D_2} - \tilde{f}\chi_C)$$

$$= \iint_R \tilde{f}\chi_{D_1} + \iint_R \tilde{f}\chi_{D_2} - \iint_R \tilde{f}\chi_C \quad (\text{by } (A))$$

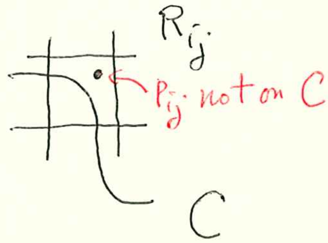
Observing that $\tilde{f}\chi_{D_1} = f$ on D_1 and $=0$ elsewhere, $\tilde{f}\chi_{D_1}$ is the universal extension of f from D_1, so

$$\iint_R \tilde{f}\chi_{D_1} = \iint_{D_1} f.$$

Similarly, $\iint_R \tilde{f}\chi_{D_2} = \iint_{D_2} f$.

We claim $\iint_R \tilde{f}\chi_C = 0$.

For, let P be a partition on R. Since $\tilde{f}\chi_C$ vanishes everywhere except possibly on the curve C, we can always pick a tag point $p_{ij} \in R_{ij}$ such that $(\tilde{f}\chi_C)(p_{ij}) = 0$, so



$$R(\tilde{f}\chi_C, P) = 0$$

Letting $\|P\| \to 0$, $\iint_R \tilde{f}\chi_C = 0$ too.

Putting things together,

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f . \quad \blacksquare$$

(Forget to point out that when f is piecewise continuous, $\tilde{f}\chi_{D_1}$, $\tilde{f}\chi_{D_2}$, $\tilde{f}\chi_C$ are piecewise continuous functions.)

(Cont'd)

We use the double integral to define area.

Let D be a region in $\mathbb{R}^2$. We define its <u>area</u> to be

$$A = \iint_D dA = \iint_R \chi_D \, dA ,$$ where R is a rectangle containing D.

Here $\iint\limits_D dA$ means $\iint\limits_D 1\, dA$, that is, the integral of the constant function 1 over D. As

$$\tilde{1}(x,y) = \begin{cases} 1, & (x,y)\in D \\ 0, & (x,y)\notin D, \end{cases}$$

$\tilde{1} = \chi_D$, so

$$\begin{aligned} \iint\limits_D dA &= \iint\limits_D 1\, dA \\ &= \iint\limits_R \tilde{1}\, dA \\ &= \iint\limits_R \chi_D\, dA, \end{aligned}$$ which justifies our notation.

Motivation of the definition: Let P be a partition on R. We let

$\mathcal{A}$ = subrectangles of P lying inside D

$\mathcal{B}$ = ... touching both D and the outside of D

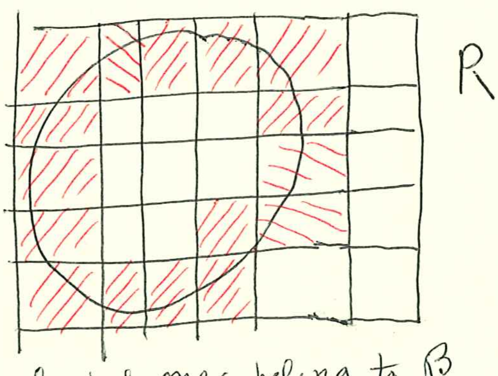
$\mathcal{C}$ = ... lying outside of D.

The outer approximate area of P

$$\sum_{\mathcal{A}} \Delta x_i \Delta y_j + \sum_{\mathcal{B}} \Delta x_i \Delta y_j.$$

The inner approximate area of P

$$\sum_{\mathcal{A}} \Delta x_i \Delta y_j.$$



shaded ones belong to $\mathcal{B}$

Theorem As $\|P\| \to 0$, both the outer approximate area and the inner approximate area converge to

$$\iint_R \chi_D .$$

Pf. As $\|P\| \to 0$, the Riemann sums

$$S(\chi_D, P) \to \iint_R \chi_D .$$

For $R_{ij} \in \mathcal{A}$, $\chi_D \equiv 1$, $\chi_D(P_{ij}) = 1$ for any tag P_{ij}.

$R_{ij} \in \mathcal{B}$, we can choose a tag P_{ij} s.t. $\chi_D(P_{ij}) = 1$

$R_{ij} \in \mathcal{C}$, $\chi_D \equiv 0$ there, so $\chi_D(P_{ij}) = 0$ for any tag.

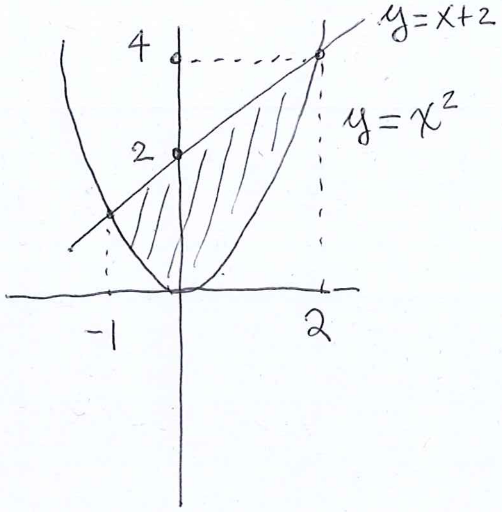
$$\begin{aligned} S(\chi_D, P) &= \sum \chi_D(P_{ij}) \Delta x_i \Delta y_j \\ &= \sum_{\mathcal{A}} \chi_D(P_{ij}) \Delta x_i \Delta y_j + \sum_{\mathcal{B}} \chi_D(P_{ij}) \Delta x_i \Delta y_j \\ &\quad + \sum_{\mathcal{C}} \chi_D(P_{ij}) \Delta x_i \Delta y_j \\ &= \sum_{\mathcal{A}} \Delta x_i \Delta y_j + \sum_{\mathcal{B}} \Delta x_i \Delta y_j + 0 \\ &= \text{outer approximate area}. \end{aligned}$$

On the other hand, we can choose tag $P_{ij} \in R_{ij}$ s.t. $\chi_D(P_{ij}) = 0$, so

$$\begin{aligned} S(\chi_D, P) &= \sum_{\mathcal{A}} \chi_D(P_{ij}) \Delta x_i \Delta y_j + \sum_{\mathcal{B}} 0 \, \Delta x_i \Delta y_j + \sum_{\mathcal{C}} 0 \, \Delta x_i \Delta y_j \\ &= \sum_{\mathcal{A}} \Delta x_i \Delta y_j \\ &= \text{inner approximate area}. \end{aligned}$$

In both cases, $S(\chi_D, P) \to \iint_R \chi_D$.

e.g. Find the area of the region enclosed by $y = x^2$, and $y = x+2$.



$$\text{area} = \iint_D dA$$

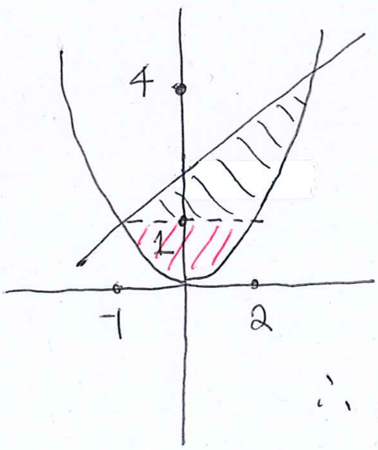
$$= \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx$$

$$= \int_{-1}^{2} (x+2-x^2)\,dx$$

$$\vdots$$

$$= \frac{1}{6}.$$

Or, consider the following decomposition : $D = D_1 \cup D_2$



D_1 : $0 \le y \le 1$, $-\sqrt{y} \le x \le \sqrt{y}$

D_2 : $1 \le y \le 4$, $y-2 \le x \le \sqrt{y}$

$$\therefore \text{area} = \iint_{D_1} dA + \iint_{D_2} dA$$

$$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy$$

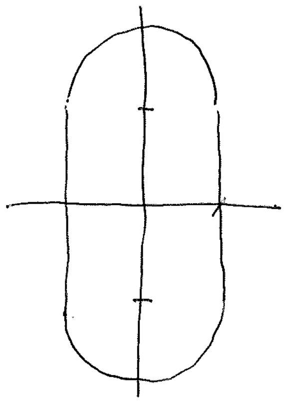
$$\vdots$$

$$= \frac{1}{6}.$$

Obviously, the first approach is simpler.

e.g. Find the area of the playing field :

$$-2 \le x \le 2, \quad -1-\sqrt{4-x^2} \le y \le 1+\sqrt{4-x^2}.$$



$$\text{area} = \int_{-2}^{2} \int_{-1-\sqrt{4-x^2}}^{1+\sqrt{4-x^2}} dy\,dx$$

$$= 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\,dx \quad \text{(take adv. of the symmetry of } D\text{)}$$

$$= 4\int_0^2 \left(1+\sqrt{4-x^2}\right)dx$$

$$= 4\left(x + \frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right)\Big|_0^2 \quad \text{(from Integral Table)}$$

$$= 4\left(2+0+2\frac{\pi}{2}-0\right) = 8+4\pi.$$

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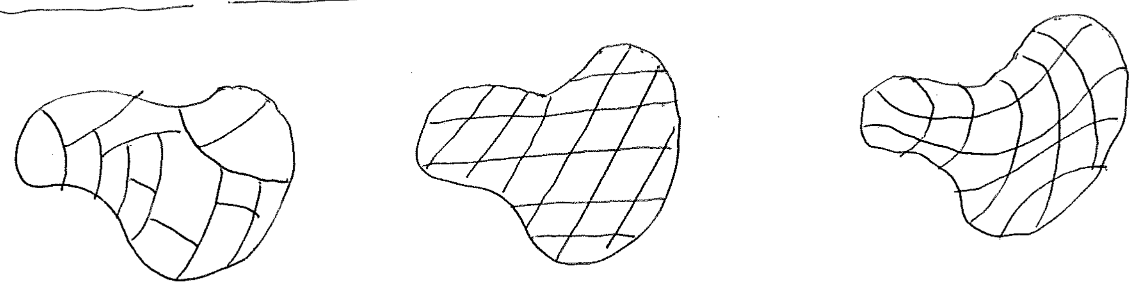
Generalized Partition



Use curves to divide a region D into finitely many sub-regions D_k, $k=1,\dots,N$.

$$\|P\| = \max\{\text{diameters of each } D_k\}$$

Generalized Riemann Sum of f

$$S(f,P) = \sum_{k=1}^{N} f(P_k)\,|D_k|, \qquad P_k \in D_k \text{ tag}, \quad |D_k| \text{ area of } D_k.$$